Sheaves of modules (Har II 5, Shaf VI 3.1)

Let $(X, \mathcal{O}_X)$ be a ringed space.

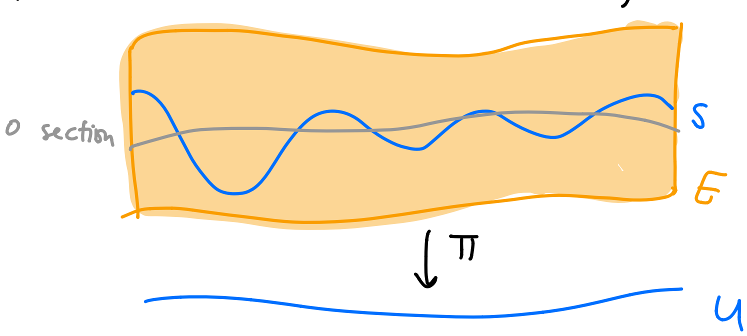
Def: A <u>sheaf of $\mathcal{O}_X$-modules</u> (or an <u>$\mathcal{O}_X$-module</u>) is a sheaf $\mathcal{F}$ s.t. for each open $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, such that for $V \subseteq U$, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible w/ the restriction $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

That is, $\mathcal{F}(V)$ is naturally an $\mathcal{O}_X(U)$-module, and $\mathcal{F}(U) \to \mathcal{F}(V)$ should be an $\mathcal{O}_X(U)$-module homomorphism.

<u>(Non-AG) Ex</u>: Let X be a (smooth, real) manifold, which is a ringed space w/ the sheaf of smooth $\mathbb{R}$-valued functions. A vector bundle on X of rank n $\mathcal{F}$ has sections over $U \subseteq X$ the set of smooth functions

$$s : U \longrightarrow E$$

where E is the total space of $\mathcal{F}$ over U (locally looks like $U \times \mathbb{R}^n$) such that $\pi \circ s = id$, where π is the projection $E \to U$



$\mathcal{F}$ is a C^∞-module as follows:

If $f: U \to \mathbb{R}$ is in $C^\infty(U)$, it acts on $s: U \to E$ in $\mathcal{F}(U)$ as $f(u)\,s(u)$.

What are the stalks of $\mathcal{F}$? We can choose $U \ni p$ s.t $E = U \times \mathbb{R}^n$. So that $s \in \mathcal{F}_p$ is represented by

$$(U,\ s: U \to \mathbb{R}^n).$$

s is C^∞ in each coordinate, so $\mathcal{F}_p$ is a free C^∞_p-module of rank n.

Def: A morphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that for each open $U \subseteq X$, $\mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism.

An $\mathcal{O}_X$-module $\mathcal{F}$ is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is locally free if X can be covered by open U s.t. $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. The rank of $\mathcal{F}$ on U is the # of copies of $\mathcal{O}_X$ needed. (If X is connected, the rank is the same everywhere.)

A locally free sheaf of rank one is called an invertible sheaf.

A sheaf of ideals on X is a sheaf of modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$. i.e. for $U \subseteq X$ open

$$\mathcal{I}(U) \subseteq \mathcal{O}_X(U) \text{ is an ideal.}$$

Push-forward and pull-back

Let $f: X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ any sheaf on X, and $\mathcal{G}$ any sheaf on Y. Recall that $f^{-1}\mathcal{G}$ is a sheaf on X defined to be the sheafification of

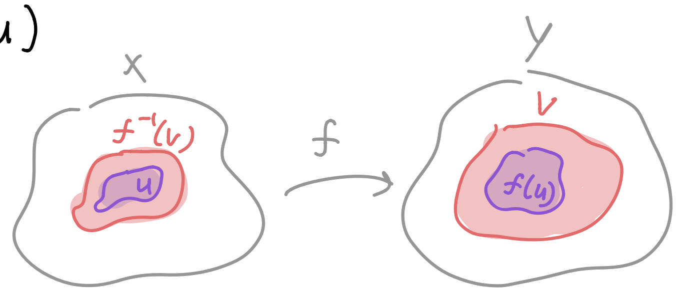
$$U \longmapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V).$$

There is a natural map

$$f^{-1} f_* \mathcal{F} \longrightarrow \mathcal{F}$$

defined $\varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$

via restriction, which factors through sheafification



We also have a natural map

$$\mathcal{G} \longrightarrow f_* f^{-1} \mathcal{G}$$

defined

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r] \arrow[dr, "\text{restriction}"'] & (f^{-1}\mathcal{G})(f^{-1}(U)) \\ & \varinjlim_{V \supseteq f(f^{-1}(U))} \mathcal{G}(V) \arrow[u, "\text{sheafification}"'] \end{tikzcd}$$

These maps give a natural bijection of sets

$$\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

If $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$, we get

$$\mathcal{G} \to f_* f^{-1}\mathcal{G} \xrightarrow{f_*\varphi} f_*\mathcal{F},$$

and if $\varphi : \mathcal{G} \to f_*\mathcal{F}$, we get

$$f^{-1}\mathcal{G} \to f^{-1}f_*\mathcal{F} \to \mathcal{F}.$$

f^{-1} is called a <u>left adjoint</u> of f_* and f_* a <u>right adjoint</u> of f^{-1}!

Now, let $f : X \to Y$ be a morphism of ringed spaces.

If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module:
$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ and $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$.

Thus, $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ gives $f_*\mathcal{F}$ the structure of an $\mathcal{O}_Y$-module. $f_*\mathcal{F}$ is called the <u>direct image</u>, or <u>pushforward</u> of $\mathcal{F}$.

If $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. Because of the adjoint property of f^{-1}, we get a morphism

$$f^{-1}\mathcal{O}_Y \to \mathcal{O}_X,$$ and we define

the pull-back (or inverse image) of $\mathcal{G}$ to be

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

(which is the sheafification of the corresponding presheaf).

Just as above, we can show that f_* and f^* are adjoint functors between $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules. That is, there's a natural isomorphism (check!)

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \tilde{\mathcal{F}}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\tilde{\mathcal{F}}).$$

Sheaves of Modules on Spec A

Let A be a ring and M an A-module. We construct the sheaf associated to M on $\mathrm{Spec}\,A$, denoted $\tilde{M}$ as follows:

On a basic open set $D(f)$, set $\tilde{M}(D(f))$ to be the A_f-module $M_f := M[1/f]$. (Thus, $\tilde{M}(\mathrm{Spec}\,A) = M$.)

Just as w/ $\mathcal{O}_{\mathrm{Spec}\,A}$, we extend this to any open set via the inverse limit:

$$\tilde{M}(U) := \varprojlim_{D(f)\subseteq U} M_f \subseteq \prod_{D(f)\subseteq U} M_f$$

Claim:

a.) $\tilde{M}$ is a sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-modules.

b.) The stalk at P is $\tilde{M}_p = M_p = M \otimes_A A_p$.

a.) is true essentially by construction. The proofs are just like those for $\mathcal{O}_{\operatorname{Spec} A}$.

The map $M \longrightarrow \tilde{M}$ is a functor. Moreover, it satisfies the following:

i.) It is exact; i.e. it takes exact sequences of A-modules to exact sequences of $\mathcal{O}_X$-modules where $X = \operatorname{Spec} A$ (Check this at stalks – localization is exact.)

ii.) It is fully-faithful: $\operatorname{Hom}_A(M, N) = \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$.
$\left((M \to N) \mapsto (\tilde{M} \to \tilde{N}) \text{ and } (\tilde{M} \to \tilde{N}) \mapsto (\Gamma(\tilde{M}) \to \Gamma(\tilde{N})) \right)$

iii.) It commutes w/ tensor product + direct sum (since these commute w/ localization). i.e.

$$\widetilde{M \otimes_A N} \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \quad \text{and} \quad \widetilde{(\oplus M_i)} \cong \oplus \tilde{M}_i .$$

iii.) If $A \to B$ is a ring homomorphism and $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ the corresponding morphism, Let M be an A-module, N a B-module. Then

$$f_*(\tilde{N}) \cong \tilde{N'},$$

where $N' = N$ considered as an A-module, and

$$f^*(\tilde{M}) = (\widetilde{M \otimes_A B}).$$

Both of these follow from def of f_*, f^* (check!).

Ex: Let $M = k[x]/(x)$ as a $k[x]$-module. Then $\tilde{M}$ is an $\mathcal{O}_X$-module on $\mathbb{A}^1_k$.

If $U = D(f)$, then

$$\tilde{M}(U) = \left(k[x]/(x)\right)_f = \begin{cases} 0 & \text{if } f \in (x), \text{ i.e. } (x) \notin U \\ k[x]/(x) \cong k & \text{if } (x) \in U \end{cases}$$

The stalks of $\tilde{M}$ are

$$\tilde{M}_p = \left(k[x]/(x)\right)_p = k[x]/(x) \text{ if } p = (x), \ 0 \text{ otherwise.}$$

i.e. $\tilde{M}$ is a skyscraper sheaf at the origin.

(Notice that $\mathrm{Supp}(M)$ consists of the points at which the stalk is nonzero.)

Just as the structure sheaf on $\mathrm{Spec}\,A$ is the building block for structure sheaves on schemes, the sheaves $\tilde{M}$ are building blocks for quasi-coherent sheaves, which we will see in the following section.